Investigating the physical nature of the Coriolis effects in the fixed frame

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It is shown that investigating Coriolis effects in the fixed frame leads to a clearer understanding of their physical nature: the body carries along its initial velocity, larger or smaller than the velocity of its corresponding point during the motion. A simple but thorough calculation of some effects (the Foucault pendulum and the eastward drift of a falling body) is given.

INTRODUCTION

The study of the laws of addition of velocities in a rotating frame leads, by derivation with respect to time, to the well-known Coriolis acceleration, and later to the need of a Coriolis inertial force when computing a problem in a rotating frame. This is the usual way of acquainting students with the so-called Coriolis effects, such as the rotation of the Foucault pendulum and the eastward deviation of a falling body.[1-3]

However, the physical reason for the effects is hidden by such a presentation, which usually leads to computing them in the rotating frame. Among standard textbooks, only Sommerfeld[4] initiates a discussion in the absolute frame, but his calculations are somewhat lengthy.

In Sec. I, we illustrate with a simple example, and by staying in the fixed frame, that the initial velocity carried by a moving body is responsible for Coriolis effects. In Sec. II, we show that an observation in the fixed frame of both a falling body and the Foucault pendulum yields immediate understanding of the physical meaning of the effects. In Sec. III, we show that this investigation eases the computation of the effects.

One of us (J.R.) is a Professor of Physics in a French "taupe," a high school section paralleling the first years of the University; the other author (E.O.) has been a Teaching Assistant for three years with the Ecole Normale Supérieure, in a section preparing students for the Agrégation, a degree in physics education. Both of us introduced this presentation of Coriolis effects to our students, who showed great interest in it, and seemed to master the physical phenomena better.

In all the examples of this paper, we shall use the Earth as a rotating frame. We shall study the motion in a frame which coincides with the (east, north, up) frame of the starting point, but which remains fixed during the motion, thus being Galilean.

I. TRAVELING NORTH IN THE NORTHERN HEMISPHERE

In this section, we consider a particle traveling north along a meridian, in the northern hemisphere, at velocity v_N, as shown in Fig. 1. Assuming that the motion is initiated at latitude λ, the initial velocity of the particle has then an eastward component in the fixed frame:

$$v_E = R\omega \cos\lambda, \tag{1.1}$$

where R and ω are the radius and rotational velociy of the Earth, respectively. The particle then moves from latitude λ to latitude $\lambda + d\lambda$ in a time

$$t = R\, d\lambda / v_N. \tag{1.2}$$

During this motion, v_E remains a constant and the eastward displacement of the particle is

$$x_E = v_E t = R^2\omega \cos\lambda \, d\lambda / v_N. \tag{1.3}$$

Meanwhile, the eastward drift of the would be destination point has only been

$$x_E' = R^2\omega \cos(\lambda + d\lambda)\, d\lambda / v_N. \tag{1.4}$$

Therefore, in the absence of any force restricting the particle on a meridian, we should observe an eastward drift in the rotating frame:

$$X = x_E - x_E' = R^2\omega \sin\lambda (d\lambda)^2 / v_N. \tag{1.5}$$

If, on the contrary, we now want the particle, of mass m, to stay on the meridian, we should apply on it a westward

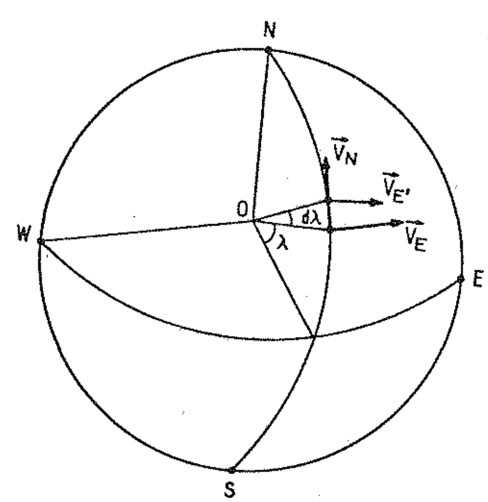

Fig. 1. Traveling north in the northern hemisphere.

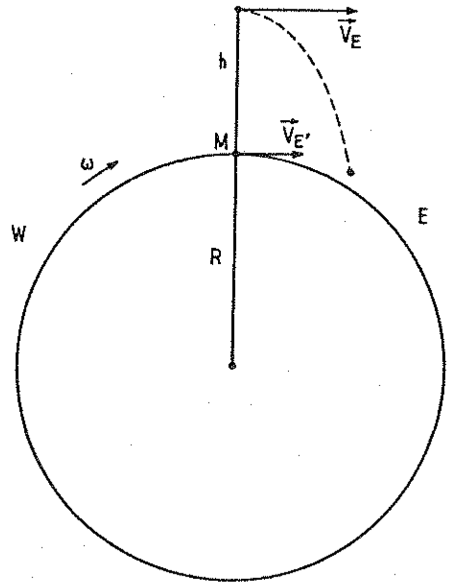

Fig. 2. Eastward displacement during free fall in the plane of the equator.

force, f_W, able to cancel this drift during the time t, given by (1.2):

$$f_W = 2m\omega v_N \sin\lambda, \qquad (1.6)$$

which is the classical expression of the reaction force of a rail against a northbound train.

This simple example illustrates that the origin of the Coriolis effects lies in the initial velocity in the absolute frame, which the moving body will keep during its motion, regardless of its position in the relative frame. The fact that it carries along its initial velocity can be regarded as a trivial statement from classical mechanics. However, it explains the true nature of Coriolis effects.

II. MORE CORIOLIS EFFECTS: THE EASTWARD DRIFT OF THE FALLING BODY AND THE FOUCAULT PENDULUM

(a) In the case of free fall, Fig. 2 shows in a very simple way the reason for the eastward drift.

We assume the experiment to be performed at the equator, which is the circle in Fig. 2. The absolute initial velocity of the falling body is eastward, and its value is

$$v_E = (R + h)\omega, \qquad (2.1)$$

which is greater than the velocity of point M on the Earth,

$$v_E' = R\omega. \qquad (2.2)$$

Accordingly, during the motion, the falling body "overtakes" its would be destination point, and an eastward drift is observed. The nonuniformity of g in the fixed frame has the effect of reducing v_E during the fall, and the computation must proceed with caution, as will be shown in Sec. III.

(b) In the case of the Foucault pendulum we shall assume that the initial position of the pendulum is south from its suspension point because, in this case, we find ourselves again with the problem in Sec. I. Figure 3 shows the trajectory of the pendulum, its suspension point, and its departure and destination points, in the fixed frame, during the first half-period of the motion. It is easy, in this figure, to notice the tilt of the north–south direction in the fixed frame. This is the reason for the angle α, which illustrates the rotation of the pendulum.

As in Sec. II(a), one must be cautious with the computation, because of the existence of a westward component of the tension of the suspension string, which decreases v_E during the motion, as will be shown in Sec. III. Nevertheless, the reason for the rotation of the Foucault pendulum is the different initial velocities of the pendulum and suspension point. When the initial position is north the effect is opposite but α keeps its sign, whereas when the initial position is west (or east), one easily shows a north (or south) deviation by a study in the plane of the great circle tangent to the local east–west line (these directions are for the northern hemisphere).

III. THOROUGH CALCULATIONS OF THE EFFECTS

(a) *Eastward drift of the falling body*. Figure 4 is drawn in the plane of the great circle tangent to the east–west line, at latitude λ. The initial velocity of the falling body in the fixed frame is

$$v_E = (R + h)\omega \cos\lambda. \qquad (3.1)$$

We must take into account the westward component of g at point P during the motion, the value of which is approximately

$$g_x = -gx/R \qquad (3.2)$$

in the fixed frame.

This yields an equation of motion along direction x:

$$\ddot{x} + gx/R = 0, \quad x(0) = 0, \dot{x}(0) = v_E. \qquad (3.3)$$

Hence,

$$x = (R/g)^{1/2} v_E \sin[(g/R)^{1/2} t], \qquad (3.4)$$

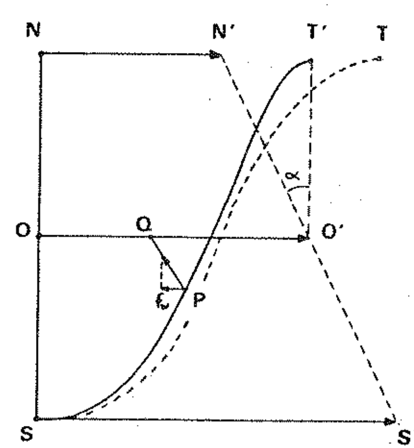

Fig. 3. Scheme of trajectories of the Foucault pendulum and suspension point during an oscillation. SS', OO', and NN' are displacements of initial position, suspension point, and destination point. ST is an idealized trajectory for the pendulum, not taking f_W into account. PQ is the projection of the suspension string at time t, clearly showing the westward component f_W to the tension. The result of the action of f_W is a bent from ST to ST', and therefore a decrease in α.

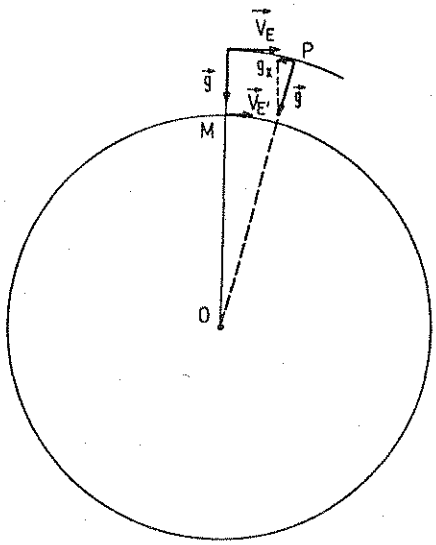

Fig. 4. Complete investigation of the free fall, in the plane of the great circle tangent to the parallel, at latitude λ, showing component g_x, reducing v_E during the fall.

and for small t

$$x = (R/g)^{1/2} v_E[(g/R)^{1/2} t - (1/6)(g/R)^{3/2} t^3]. \quad (3.5)$$

The absolute eastward drift is then obtained for $t = (2h/g)^{1/2}$:

$$x = v_E t - (1/6) v_E (g/R) t^3, \quad (3.6)$$

whereas the drift of point M is only

$$x' = v_E' t = R\omega\,(\cos\lambda) t. \quad (3.7)$$

The relative eastward drift is therefore

$$X = (2/3) h\omega \cos\lambda (2h/g)^{1/2}, \quad (3.8)$$

which is the classical result,[2] yielding a numerical value of 7 cm for a body falling from the Eiffel Tower (h = 300 m) in Paris (λ = 49°), and 15 cm from the Empire State Building (h = 448 m; λ = 41°). A computation not taking g_x into account would omit the factor 2/3.

(b) *The Foucault pendulum.* As in Sec. II, we assume that the experiment is performed in the northern hemisphere and that the initial position is south. Our purpose is to compute the angle α, shown in Fig. 3. Assuming $OS = r$ is the initial amplitude of the pendulum and a is the length of its suspension string, we write the westward component of the tension of the string on the pendulum of mass m as

$$f_W = (mg/a)(x - v_E' t). \quad (3.9)$$

Therefore, the equation of motion along x is

$$\ddot{x} = -(g/a)(x - v_E' t), \quad x(0) = 0, \dot{x}(0) = v_E, \quad (3.10)$$

which yields

$$x = v_E' t + (a/g)^{1/2}(v_E - v_E') \sin[(g/a)^{1/2} t]. \quad (3.11)$$

For a half-period of the pendulum, $t = \pi(a/g)^{1/2}$, and from (3.11), $x = v_E' t$, whereas the corresponding point on the Earth, N, has only moved to N' with $NN' = x'' = v_E'' t$. Hence,

$$\alpha = (v_E' - v_E'') t/r = \omega t \sin\lambda. \quad (3.12)$$

The period of precession of the Foucault pendulum is therefore

$$T = 2\pi/(\omega \sin\lambda), \quad (3.13)$$

which is the classical result: The Foucault pendulum rotates in one day at the poles, in 1.52 days in New York (clockwise) and Wellington (counterclockwise), in 5.75 days in Caracas, and does not rotate on the equator. A computation not taking f_W into account would lead to a period too small by 1/2.

IV. DISCUSSION AND CONCLUSION

We showed that, in the fixed frame, one easily understands the true nature of the Coriolis effects, which lies in the maintaining of initial velocities in that frame. We will show that this approach gives an explanation for the mathematical form of the Coriolis acceleration in the rotating frame.

If we want Coriolis effects to take place, we first need an inhomogeneous field of velocities of the moving frame. Therefore, it must rotate, with rotational velocity ω. Furthermore, in order to undergo Coriolis effects, we must move, in the relative frame, from one point to another where the absolute initial velocity is different. So we need a relative velocity $\mathbf{v}_r$. However, a displacement parallel to ω would have no effect, and one anticipates easily that Coriolis effects are to be characterized by the vector product $\omega \times \mathbf{v}_r$.

As a conclusion, we would like to give a happy consequence of the use of our approach. It cancels out any hesitation on the classical questions: Does a falling body drift eastward or westward in the southern hemisphere? Does the Foucault pendulum rotate at the pole? etc. In order to answer such questions, the only alternative to a real understanding of the physical origin of the effects would be the use of corkscrew-achieved mnemonics.

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