SYMPOSIUM
ON THE HYDROLOGY
OF GLACIERS

Cambridge, 7—13 September 1969

organized by the

GLACIOLOGICAL SOCIETY

CAN A WATER-FILLED CREVASSE REACH THE BOTTOM SURFACE OF A GLACIER?

By J. WEERTMAN

(Department of Materials Science, Department of Geological Sciences, and Materials Research Center, Northwestern University, Evanston, Illinois 60201, U. S. A.)

ABSTRACT. Nye has estimated that the depth L of crevasses is equal to $T/\varrho g$, where T is the tensile stress causing extending flow, ϱ is the density of ice, and g is the gravitational acceleration. This expression for L is derived on the assumption that the crevasses are closely spaced and free of water. It is shown in this paper that the depth of an isolated crevasse is a factor $\pi/2$ greater than the depth calculated by Nye for closely spaced crevasses. It is shown further that the presence of water in a crevasse can increase its depth. A crevasse filled with water up to at least 97.4% of its depth can penetrate to the bottom of a glacier. Water-filled cavities can exist directly beneath water filled crevasses.

RÉSUMÉ. *Une crevasse emplie d'eau peut-elle atteindre la surface basale d'un glacier?* Nye a estimé que la profondeur L des crevasses est égale à $T/\varrho g$, où T est la contrainte d'élongation causant un écoulement d'extension, est la densité de la glace et g est l'accélération de la pesanteur. Cette expression pour L est dérivée en supposant que les crevasses sont proches et vides d'eau. On montre ici que la profondeur d'une crevasse isolée est plus grande d'un facteur de $\pi/2$ que la profondeur calculée par Nye pour les crevasses proches l'une de l'autre. Il est montré de plus que la présence d'eau dans une crevasse peut augmenter sa profondeur. Une crevasse emplie d'eau d'au moins 97,4% de sa profondeur peut pénétrer jusqu'au fond d'un glacier. Des cavités pleines d'eau peuvent exister directement sur des crevasses emplies d'eau.

INTRODUCTION

Nye (1955) has considered the depth of penetration of crevasses in a glacier. Figure 1 shows a cross-section of a crevassed region. It is assumed that an average tensile stress T acts across a vertical section through the glacier. Nye assumed that a crevasse, in a field of closely spaced crevasses, penetrates to that depth at which the over-burden pressure equals the tensile stress. If y is the depth in the glacier measured from the top surface, ϱ is the density of ice, and g is the gravitational acceleration, the over-burden pressure equals $\varrho g y$. The depth L of crevasse penetration thus is equal to $L = T/\varrho g$.

Nye's method of estimating crevasse depth ignores the effect of the stress concentrations that can exist in the immediate vicinity of the tip of a crevasse. It is reasonable to ignore stress concentrations for crevasses in a field of closely spaced crevasses because of the "blunting" effect that neighboring crevasses have on any individual crevasse. (The stress component that can exist in the thin slab of ice between two closely spaced neighboring crevasses [see Fig. 1] is only the over-burden compressive stress. No tensile stress can exist within such a slab. Therefore no large stress concentrations can be set up at the tips of the crevasses.) However, for an isolated crevasse, such as is shown in Figure 2, this blunting effect does not exist. Stress concentrations must be taken into account in order to determine the
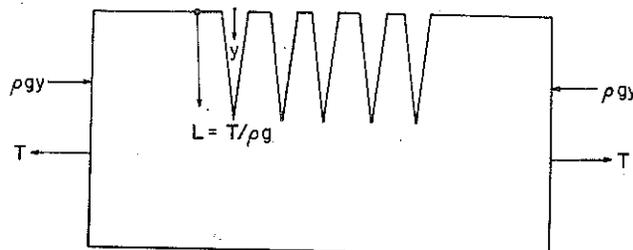

Fig. 1. Field of closely spaced crevasses.

depth of the crevasse. One of the purposes of this paper is to determine crevasse penetration depth for the situation shown in Figure 2.

Nye's estimate of crevasse depth made use of the implicit assumption that the crevasses were free of water. However if all the crevasses of Figure 1 were filled with water to the same depth the estimate of crevasse depth would be virtually unchanged. The criterion that a crevasse penetrates the glacier to a depth at which the overburden pressure equals the tensile stress T would still give $L \approx T/\varrho g$ whether the crevasse is water filled or not.

In an isolated crevasse the presence of water has a profound effect on the depth to which the crevasse can penetrate the glacier. The pressure of water pushing against the walls of the crevasse acts as a wedge force to cause further penetration into the glacier. It is the primary purpose of this paper to establish theoretically this contention.

ISOLATED WATER-FREE CREVASSE

Consider Figure 2 once more. Assume that the deformation about the crevasse immediately after it has opened up is elastic in character. (Creep deformation will be considered later in the paper.) Let the net displacement across the crevasse at a depth y be $D(y)$. (The problem we consider is two dimensional. The crevasse extends an infinite distance in the direction perpendicular to the plane of Figure 2.)
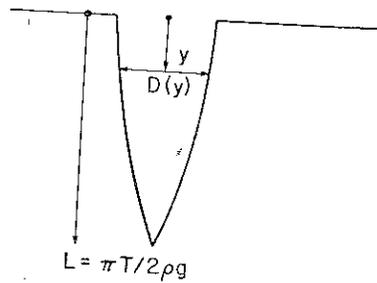
$$L = \pi T/2\rho g$$

Fig. 2. Isolated water-free crevasse.

The displacement $D(y)$ can be considered to arise from a distribution $B(y)$ of infinitesimal edge dislocations such as are shown in Figure 3. Here $B(y)\delta y$ represents the total strength of the edge dislocations lying between y and $y + \delta y$ on the plane of the crevasse. Thus $D(y) = \int_y^L B(y) \mathrm{d}y$ or $B(y) = -\mathrm{d}D/\mathrm{d}y$.

The tensile or compressive stress $\sigma_{xx}(y)$ which the dislocation distribution $B(y)$ causes to act across the plane of the crevasse is given by the following equation [see the analogous equation (4) of Kuang and Mura (1968) or equation (37) of Mura and Dundurs (1971)]

$$\sigma_{xx}(y) = (\mu/2\pi[1-\nu]) \int_{-L}^{L} B(y')(y-y')^{-1}\mathrm{d}y' + (\mu/\pi[1-\nu]) \int_0^L B(y')y'(y'-y)(y+y')^{-3}\mathrm{d}y' \quad (1)$$

where ν is Poisson's ratio, μ is the shear modulus, and $B(-y)$ is defined to be equal to $-B(y)$.

The surfaces of the crevasse are free of traction. Thus, the dislocation distribution function $B(y)$ must be such that in the region $0 \leq y \leq L$ the equation

$$\sigma_{xx}(y) + T - \varrho g y = 0 \quad (2)$$

is satisfied where $\sigma_{xx}(y)$ is given by Equation (1). That is, the sum of the stress produced by the dislocations, the tensile stress, and the hydrostatic pressure must equal zero on the crevasse faces.

Kuang and Mura (1968) have given a general method for solving integral equations of the form of Equation (2). They have given explicit solutions for problems in which the $\varrho g y$ term is

dropped from the equation. The exact solution of integral equations of the form of Equation (2) involves rather complicated expressions.

Consider for the moment a problem similar to but simpler than the crevasse problem. The problem of the shear crack near a free surface (which involves screw dislocations) has a simpler integral equation than Equation (1). The equation which replaces Equation (1) for this problem is (Weertman, 1964)

$$\sigma_{x\partial}(y) = (\mu/2\pi) \int_{-L}^{L} B(y')(y - y')^{-1}\mathrm{d}y' \qquad (3)$$

in which $\sigma_{x\partial}(y)$ is the shear stress acting across the plane. If the friction stress σ_f which prevents sliding across the shear-crack face increases linearly with distance from the free surface so that $\sigma_f = \sigma_0 y/y^*$, where σ_0 and y^* are constants, the equation which replaces Equation (2) is

$$\sigma_{x\partial}(y) + S - (\sigma_0/y^*)y = 0 \qquad (4)$$

where S is the external applied shear stress. A relatively simple, but exact, solution exists for this integral equation (Weertman, 1964).

Physically it is to be expected that the solution of the shear crack problem (that is, of Equation (4)) is qualitively the same as the solution of the tensile crack problem (Equation (2)). The solutions would be identically the same, apart for the constant term $(1 - \nu)$, were it not for the second integral which has the limits of 0 and L, in Equation (1).

Kuang and Mura (1968) considered both problems involving edge dislocations and problems involving screw dislocations. In the case of the problems they solved exactly, the solutions for edge dislocations were virtually the same as the solutions for screw dislocations, apart from a constant factor analogous to $(1 - \nu)$. (Compare their results plotted in their figures 5 and 6.) We believe, therefore, that an approximate but reasonable solution to Equation (2) can be obtained by dropping the second integral of Equation (1). (The Appendix gives an estimate of the error involved in making this approximation.)

From the known exact solution of Equation (4) (see p. 1045 of Weertman, 1964) we obtain the following approximate solution of Equation (2):
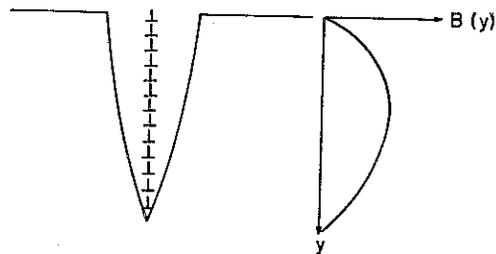

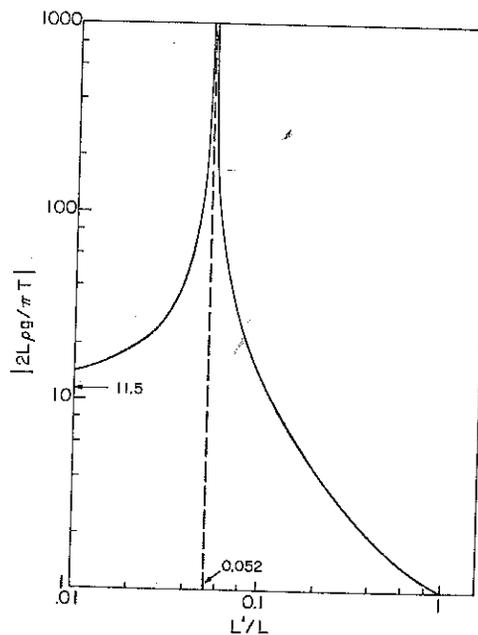Fig. 3. Distribution of infinitesimal edge dislocations which describe an opened crevasse.

Fig. 4. Plot of $|2L\rho g/\pi T|$ versus L'/L calculated from Equation (8) with $\rho/\rho' = 0.92$. The quantity $(2L\rho g/\pi T)$ is positive for $L'/L > 0.052$. It is negative for $0 \leq L'/L < 0.052$.

$$B(y) = (4[1 - \nu]\rho g y/\pi\mu)\log|\{L + (L^2 - y^2)^{1/2}\}/y| \qquad (5)$$
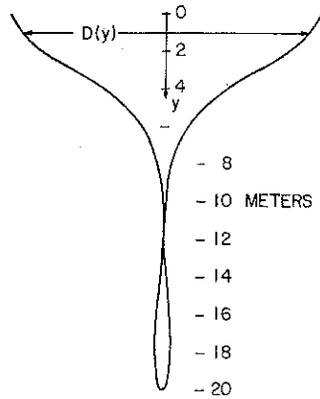
Fig. 5. Elastic displacement $D(y)$ versus depth y for a crevasse of depth $L=20$ m, $L'=5$ m, $T=0.38$ bar, $\varrho/\varrho'=0.92$, and $T-T'=0.0105$ bar. The displacement at $y=0$ is approximately 0.057 mm.
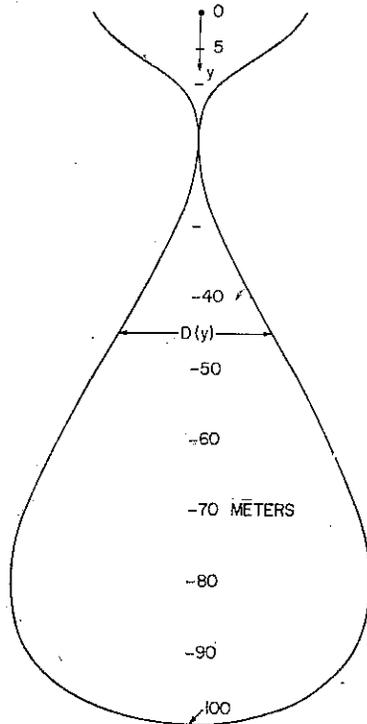

Fig. 6. Elastic displacement $D(y)$ versus depth y for a crevasse of depth $L=100$ m, $L'=10$ m, $T=0.64$ bar, $\varrho/\varrho'=0.92$, and $T-T'=0.188$ bar. The displacement at $y=80$ m is approximately 0.3 mm.

The depth L is given by

$$L = \pi T/2\varrho g. \tag{6}$$

Equation (6) supplies the answer to one question sought in this investigation. It shows that the depth of an isolated, water-free crevasse is a factor $\pi/2$ greater than the depth of closely spaced crevasses determined by Nye.

WATER-FILLED CREVASSE

Consider next an isolated crevasse which is filled with water from $y=L$ to $y=L'$, where $L' < L$. The faces of the crevasse no longer are traction free. At depths $L' \leq y \leq L$ a normal compressive stress $\varrho'g(L'-y)$ is exerted on the crevasse walls, where ϱ' is the density of water. Equation (2) is still valid for $0 \leq y \leq L'$, but for $L' \leq y \leq L$ it must be replaced with

$$\sigma_{xx}(y) + T - \varrho gy - \varrho'g(L'-y) = 0. \tag{7}$$

Equations (2) and (7) can be solved by using the general solution given in Weertman (1964, p. 1037). The solution for L is found easily from equation* (6) of Weertman (1964). It is given by the equation

$$L - (\pi T/2\varrho g) + L'(\varrho'/\varrho)[(\pi/2) - \sin^{-1}(L'/L)] -$$
$$- (\varrho'/\varrho)(L^2 - L'^2)^{\frac{1}{2}} = 0. \tag{8}$$

Figure 4 shows a plot of the logarithm of $(2L\varrho g/\pi T)$ versus L'/L given by Equation (8) calculated for $\varrho/\varrho' = 0.92$. It can be seen that as the ratio L'/L decreases in value from 1 to 0.052 the quantity $2L\varrho g/\pi T$ increases monotonically from the value 1 to $+\infty$. For values of the ratio L'/L in the range $0 \leq L'/L < 0.052$ the term $2L\varrho g/\pi T$ is negative in value. As L'/L increases in value from 0 to 0.052 the quantity $2L\varrho g/\pi T$ decreases monotonically in value from -11.5 to $-\infty$.

The dislocation density given from the solution (found from equation [5] of Weertman [1964]) of Equations (2) and (7) for a water filled crevasse is

$$B(y) = \{2[1-v]\varrho g/\pi\mu\}\{2y \log |\{L+(L^2-y^2)^{\frac{1}{2}}\}/y| -$$
$$- (\varrho'/\varrho)y \log |\{(L^2-y^2)^{\frac{1}{2}} + (L^2-L'^2)^{\frac{1}{2}}\}/\{(L^2-$$
$$- y^2)^{\frac{1}{2}} - (L^2-L'^2)^{\frac{1}{2}}\}| + (\varrho'/\varrho)L' \log |\{y(L^2-$$
$$- L'^2)^{\frac{1}{2}} + L'(L^2-y^2)^{\frac{1}{2}}\}/\{y(L^2-L'^2)^{\frac{1}{2}} -$$
$$- L'(L^2-L'^2)^{\frac{1}{2}}\}|\} \tag{9}$$

where $0 \leqq y \leqq L$ and L is given by Equation (8). For values of $L'/L < 0.389$ Equation (9) predicts that $B(y)$, and thus also $D(y)$, is less than zero at the larger values of y. (For $0.389 < L'/L \leqq 1$, $B(y)$ always is positive.) Although mathematically correct this result is not physically plausible. A negative value of $D(y)$ implies a crevasse squeezed closed.

The solution represented by Equations (8) and (9) is that of a crevasse which is free to let its depth L adjust to any value. Suppose L and L'/L are given fixed, arbitrary values. From Figure 4 the value of T is then specified. Let T' represent this value of T. In general T' will not equal the applied tensile stress T. (For $L'/L < 0.052$ it even can be negative in value.) Suppose another distribution function $B^*(y)$ could be found which produces a constant stress $T^* = -(T - T')$ between $0 \leqq y \leqq L$. This distribution when added to that given by Equation (9) will satisfy the problem. The well-known dislocation density function

$$B^*(y) = 2(T - T')(1 - \nu)\mu^{-1}y/(L^2 - y^2)^{\frac{1}{2}}$$

(10)

(where $0 \leqq y \leqq L$) does produce the required stress.

By combining Equations (9) and (10), negative values of the total dislocation distribution function $B(y) + B^*(y)$ will not exist if $T - T'$ is made sufficiently large.

The displacement $D(y)$ found by integrating $\int_{y}^{L} [B(y) + B^*(y)]dy$ is

$$
\begin{aligned}
D(y) = & (2[1 - \nu]\varrho g/\pi\mu)[(L^2 - y^2)^{\frac{1}{2}}\{L - (\varrho'/\varrho)(L^2 - L'^2)^{\frac{1}{2}}\} - y^2 \log[\{L + (L^2 - y^2)^{\frac{1}{2}}\}/y] + \\
& + (\varrho'/2\varrho)(y^2 + L'^2) \log |\{(L^2 - L'^2)^{\frac{1}{2}} + (L^2 - y^2)^{\frac{1}{2}}\}/\{(L^2 - L'^2)^{\frac{1}{2}} - (L^2 - y^2)^{\frac{1}{2}}\}| - \\
& - yL'(\varrho'/\varrho) \log |\{y(L^2 - L'^2)^{\frac{1}{2}} + L'(L^2 - y^2)^{\frac{1}{2}}\}/\{y(L^2 - L'^2)^{\frac{1}{2}} - L'(L^2 - y^2)^{\frac{1}{2}}\}| \,] + \\
& + (2(T - T')[1 - \nu]/\mu)(L^2 - y^2)^{\frac{1}{2}}.
\end{aligned}
$$

(11)

The dislocation distribution given by Equation (10) is that associated with a crack of fixed half-width L opened up by a tensile stress $(T - T')$ in an elastic material. The tensile stress immediately ahead of such a crack is infinite. In this paper it is assumed implicitly that ice cannot support an appreciable tensile stress. Therefore the solution obtained through use of Equation (10) is not entirely satisfactory. As long as $B^*(y) > 0$ a tensile stress exists ahead of the crevasse tip which is large enough to cause further penetration of the crevasse into the glacier.

According to Equation (11) a water-filled cavity can be pinched off from a crevasse. Figures 5 and 6 show two examples. In Figure 5 the crevasse of depth $L = 20$ m is pinched off at a depth of 11.3 m. A water-filled cavity exists between $y = 11.3$ m and $y = L = 20$ m. In Figure 6 the crevasse of depth $L = 100$ m is pinched closed at $y = 18$ m. (Displacements produced by creep deformation are, of course, much larger than the elastic displacements indicated in Figures 5 and 6.)

Once a crevasse is pinched off at an intermediate depth (that is, for a value of y less than L) the general solution (equation (5) of Weertman, 1964) used to obtain Equations (9) and (10) is not valid. Another, more complicated, equation (equation (7) of Weertman, 1964) must be used to find $B(y)$ both in the region above the pinch-off depth and in the cavity below it.

Figure 7 shows a plot of the maximum permitted value of the quantity $(2L\varrho g/\pi T)$ calculated from Equation (11) versus L'/L. The term T' in Equation (11) was determined from Equation (8) (with T' substituted for T in that equation). The term $(2L\varrho g/\pi T)$ was found from Equation (11)
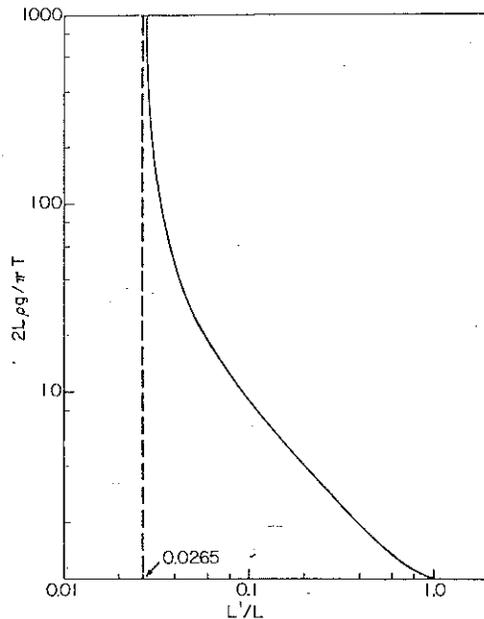
Fig. 7. Plot of $2L\rho g/\pi T$ versus L'/L obtained from Equation (11) (see text).

for the smallest value of T which ensured that $D(y) \geqq 0$ for all values of y in the range $0 \leqq y \leqq L$. Therefore the curve shown in Figure 7 represents the deepest crevasse that can exist for an applied tensile stress T at a given value of L'/L. The crevasse depth is infinite for all values of L'/L in the range $0 \leqq L'/L \leqq 0.0265$.

We conclude that a crevasse filled with water up to at least 97.35% of its depth can reach the bottom surface of a glacier. (When a crevasse is filled to at least 92% of its depth the water pressure at the bottom of the crevasse is equal to or greater than the over-burden ice pressure.)

EFFECT OF CREEP DEFORMATION

In the analysis just given it is assumed that the deformation and the stress field around a crevasse are determined solely by elastic stress- and strain-field solutions. This assumption could be true for a short time immediately after a crevasse suddenly formed.

Eventually a creep deformation would largely determine the stresses existing around a crevasse. I do not believe that the general results obtained using an elastic solution are modified significantly when creep deformation is taken into account. It has been shown (Weertman, 1969; Green and others, 1969) that the stress and strain-rate fields about a tensile or shear crack in a material obeying the creep equations of a Newtonian solid are formally identical to the elastic stress- and strain-field solutions. Thus the solution for a crevasse in a glacier which obeys the Newtonian creep equation is identical in form to what we have found using elastic equations. Of course, the creep of ice obeys the power-law creep equation rather than the linear, Newtonian creep equation. However, this fact should not cause the exact solution of the problem to be qualitatively different from the answer we have found.

SUMMARY

We have shown that the depth of an isolated, water-free crevasse is a factor of $\pi/2$ greater than the depth of a crevasse in a field of closely spaced, water-free crevasses. We also have shown that there is no limit to the depth of an isolated water-filled crevasse. There is no reason why such a crevasse may not penetrate the bottom surface of a glacier. The apparent rarity or even complete absence of crevasses that do penetrate a glacier is a real problem. Perhaps crevasses always exist in closely spaced fields and thus always are blunted. It is likely that a water-filled crevasse which does reach the bottom surface actually is pinched closed at a relatively shallow depth. A water-filled cavity beneath the pinched-off crevasse may extend to the glacier bed.

(The analysis of this paper is extended easily to the problem of a "crevasse" that starts from

the lower surface of a glacier or of a floating ice shelf. Suppose water exists at the lower surface under a pressure equal to the ice over-burden pressure. Equation (5) describes the dislocation density of the crevasse provided that y is measured upwards from the bottom surface and ϱ is replaced with $(\varrho' - \varrho)$. The length L of the crevasse is given by Equation (6), again provided that ϱ is replaced with $(\varrho' - \varrho)$. In a floating ice shelf T is equal to $1/2\ (\varrho' - \varrho)gh$ where h is the ice thickness. Thus the length L of the crevasse that starts from the bottom surface of a floating ice shelf is approximately $\pi h/4$.)

The theory of this paper has been extended to cover the problem of magma transport in the Earth's crust (Weertman, 1971). An additional result from this extended theory which is applicable to crevasses in glaciers is: a pinched-off water-filled cavity should be able to break-off from a crevasse and descend to the bottom of a glacier.

ACKNOWLEDGEMENT

REFERENCES

GREEN, W. V., *and others*. 1969. Effect of cracks on creep rate—part II (experimental), by W. V. Green, E. G. Zukas and J. Weertman. *Transactions Quarterly of American Society for Metals*, Vol. 62, No. 2, p. 512–20.

KUANG, J. G., *and* MURA, T. 1968. Dislocation pile-up in two-phase materials. *Journal of Applied Physics*, Vol. 39, No. 1, p. 109–20.

MURA, T., *and* DUNDURS, J. 1971 Deformation, modes and microstructure. (*In* Te'eni, M., *ed. Proceedings of International Conference on Structure, Solid Mechanics and Engineering Design in Civil Engineering Materials, University of Southampton, April 21–25, 1969.* New York, John Wiley and Sons, Inc., p. 3–12.)

NYE, J. F. 1955. Comments on Dr. Loewe's letter and notes on crevasses. *Journal of Glaciology*, Vol. 2, No. 17, p. 512–14. (Letter.)

WEERTMAN, J. 1964. Continuum distribution of dislocations on faults with finite friction. *Bulletin of the Seismological Society of America*, Vol. 54, No. 4, p. 1035–58.

WEERTMAN, J. 1969. Effect of cracks on creep rate. *Transactions Quarterly of American Society for Metals*, Vol. 62, No. 2, p. 502–11.

WEERTMAN, J. 1971 Theory of water-filled crevasses in glaciers applied to vertical magma transport beneath oceanic ridges. *Journal of Geophysical Research*, Vol. 76, No. 5, p.1171–83.

APPENDIX

A more exact solution of Equation (2) than was obtained in the text can be found by inserting Equation (5), the approximate solution for $B(y)$, into the second integral (the integral with the limits of 0 and L) of Equation (1). The new value of $\sigma_{xx}(y)$ then can be used in Equation (2) and a more exact value of $B(y)$ and L sought. This process then can be repeated to find still more exact solutions of Equation (2).

If Equation (5) is inserted into the second integral of Equation (1) and if this integral is evaluated numerically, one obtains the contribution $\sigma_{xx}^*(y)$ to the stress $\sigma_{xx}(y)$

$$\sigma^*{}_{xx}(y) \simeq (4\varrho g L_0/\pi^2)[c_1 \exp(-y/y_0) - c_2] \tag{A1}$$

where $L_0 = (\pi T/2\varrho g)$ is the value of L given by Equation (6), $c_1 = 1.59$, $c_2 = 0.03$, and $y_0 = 0.0963\ L_0$.

The use of Equation (6) of Weertman (1964) and Equation (A1) leads to the next approximation for L:

$$L \simeq L_0[1 + (2/\pi)^2\{(c_1 y_0/L_0) - c_2\}] = 1.05 L_0. \tag{A2}$$

Thus the neglect of the second integral of Equation (1) appears to introduce only about a 5% error in the determination of the depth of a crevasse.