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DEFORMATION OF FLOATING ICE SHELVES

By J. Weertman

(Naval Research Laboratory, Washington, D.C.)

Abstract. The problem of the creep deformation of floating ice shelves is considered. The problem is solved using Glen's creep law for ice and Nye's relation of steady-state creep (the analogue of the Lévy-Mises relation in plasticity theory). Good agreement is obtained between an observed creep rate at Maudheim in the Antarctic and that predicted from the results of creep tests made by Glen.

Zusammenfassung. Das Problem der Kriechdeformation einer schwimmenden Eisplatte wird mit Hilfe von Glen's Eiskriechgesetz und Nye's Gleichung für den Kriechgleichgewichtszustand gelöst (Nye's Gleichung ist der Lévy-Mises Gleichung in der Plastizitätstheorie analog.). Auf diese Weise wird die in Maudheim in Antarktika beobachtete Kriechgeschwindigkeit mit der von den Glen'schen Experimenten zu erwartenden in Einklang gebracht

Introduction

The problem of the flow of ice in glaciers and ice caps has been treated by Nye in a series of very illuminating papers[1, 2, 3, 4, 5]. One problem that has not been analyzed by his methods is the creep of floating ice shelves. The analyses which have been given by Nye are inapplicable to this special problem because his solutions require a finite shear stress to exist at the bottom of the mass of ice being considered. At the bottom of a floating ice shelf the shear stress, of course, must be zero. This paper presents an analysis for this problem using Glen's creep law [6] for ice and Nye's relation [1] of steady-state creep (the analogue of the Lévy-Mises relation in plasticity theory). It turns out that the solution is almost trivial. It is equivalent to the solution of the problem of a weightless material being compressed by frictionless plates.

Theory

We consider a floating ice shelf such as is shown in Fig. 1. The shelf is assumed to be in a steady-state condition, its thickness h remaining constant in time. The thickness of the shelf is assumed to be many orders of magnitude smaller than its length. The direction y is perpendicular to the surface of the shelf, x is parallel to the surface and perpendicular to the ice front, and z is parallel both to the surface and the ice front. Consider the case where creep movement can occur in the x and y direction but not in the z direction. In the Appendix is given a case where creep in the z direction is permitted.

The equations for equilibrium are

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}&=0 \\ \frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial z}+\rho_I g&=0 \\ \frac{\partial \sigma_{zz}}{\partial z}+\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}&=0 \end{aligned} \qquad (1)$$

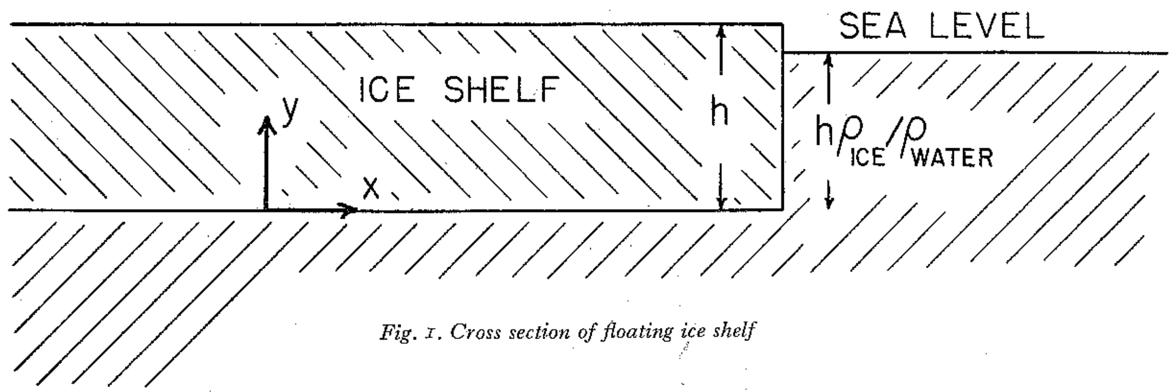

Fig. 1. Cross section of floating ice shelf

where σ_{ij} are the usual stress components (σ_{ii} is positive when in tension), g is the gravitational acceleration, and ρ_I is the density of the ice. An average density is used in the following analysis. It is a simple matter, however, to give an exact solution to the problem using a density which is a function of y.

We now make the assumption, which we believe to be quite reasonable, that at a position far from the edge of the shelf all the stress components must be independent of x and z. This assumption along with the condition that the shear stress at the top and bottom surface of the shelf must be equal to zero reduces Eqs. (1) to

$$\frac{d\sigma_{yy}}{dy}+\rho_I g=0 \qquad (2)$$

and

$$\sigma_{xy}=\sigma_{xz}=\sigma_{yz}=0 \qquad (3)$$

Integrating Eq. (2) gives

$$\sigma_{yy}=-\rho_I g(h-y) \qquad (4)$$

The stresses σ_{xx} and σ_{zz} must still be determined.

Nye's relation[1] of steady-state creep (based on Glen's creep law[6]) is

$$\dot{\epsilon}_{ij}=\lambda\sigma'_{ij} \qquad (5)$$

where

$$\lambda=A^{-n}\tau^{n-1} \qquad (6)$$

and

$$\tau^2=\tfrac{1}{2}\sigma'_{ij}\sigma'_{ij} \quad \text{(summed over } i \text{ and } j) \qquad (7)$$

and

$$\sigma'_{ij}=\sigma_{ij}-\tfrac{1}{3}\delta_{ij}\sigma_{kk} \quad \text{(summed over } k) \qquad (8)$$

Here n is a constant, A is a constant which depends on the temperature and may depend on the density, δ_{ij} is equal to zero when i does not equal j and equals one when they are equal, and $\dot{\epsilon}_{ij}$ is a strain rate which must satisfy the equation

$$\dot{\epsilon}_{ij}=\tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \qquad (9)$$

where u_i is a velocity component.

Since the creep rate is zero in the z direction the term σ'_{zz} must equal zero. This condition gives the relation that σ_{zz} is equal to $\tfrac{1}{2}(\sigma_{xx}+\sigma_{yy})$. Eq. (5) now reduces to

$$\dot{\epsilon}_{xx}=-\dot{\epsilon}_{yy}=(2A)^{-n}|\sigma_{xx}-\sigma_{yy}|^{n-1}(\sigma_{xx}-\sigma_{yy}) \qquad (10)$$

$$\dot{\epsilon}_{zz}=\dot{\epsilon}_{xy}=\dot{\epsilon}_{xz}=\dot{\epsilon}_{yz}=0 \qquad (11)$$

These equations are consistent with Eq. (9) only if $\dot{\epsilon}_{xx}$ and $\dot{\epsilon}_{yy}$ do not depend on y.

Setting $\dot{\epsilon}_{xx}$ equal to K, the creep rate, and substituting Eq. (4) into Eq. (10) one obtains

$$\sigma_{xx}=\pm 2A|K|^{1/n}-\rho_I gh+\rho_I gy \qquad (12)$$

The plus sign is used when K is positive, an expanding shelf; the minus sign when K is negative, a shrinking shelf. Both the stresses σ_{xx} and σ_{yy} are functions of y. For an expanding shelf the values of σ_{xx} are such that it is in tension at the top of the shelf and in compression at the bottom. The stress σ_{yy} is always in compression.

We now take advantage of the fact that the shelf is floating in the sea. Let ρ_W be the density of the sea water. Then σ_{xx} must obey the equation

$$\int_0^h \sigma_{xx}dy=-\int_0^{h\rho_I/\rho_W}\rho_W g(h\rho_I/\rho_W-y)dy \qquad (13)$$

Setting σ_{xx} given by Eq. (12) into Eq. (13) one obtains for K

$$K=(\tfrac{1}{4}\rho_I gh^2)^n(1-\rho_I/\rho_W)^n\bigg/\left[\int_0^h A\,dy\right]^n \qquad (14)$$

This equation gives the result that we are seeking, namely, the creep rate of the shelf. It is a particularly simple result; the creep rate in the shelf from top to bottom is constant. It turns out that K can only be positive. Therefore a shrinking shelf cannot exist.

The solution just found for the stresses and creep rate in an ice shelf applies only far from the edges of the shelf. Near an edge the stress distribution must change. Since

$$\int_0^{h\rho_I/\rho_W}\rho_W g(h\rho_I/\rho_W-y)y\,dy+\int_0^h\sigma_{xx}y\,dy$$

(the torque exerted across a unit cross-section of the shelf, perpendicular to the surface and parallel to the ice front) is not equal to zero, a torque of this magnitude must exist at either end of a shelf in order to maintain mechanical equilibrium. Such a torque may be maintained in several ways such as, for example, by means of an overhanging cliff at the ice front of the shelf.

COMPARISON WITH EXPERIMENT

We wish to compare now an observed creep rate of the ice shelf at Maudheim with that predicted by Eq. (14). Schytt[7] states that a two kilometer base line stretched at a rate of 30 cm./month. This speed gives a creep rate of $1{\cdot}8\times10^{-3}$ years^{-1}. The shelf at Maudheim was approximately 185 m. thick. The temperature of the shelf varied from $-17{\cdot}5°$ C. near the top surface to $-16{\cdot}5°$ C. at a depth of 100 m. Temperatures below 100 m. were not known. Presumably the temperature at the bottom surface is about $-1{\cdot}5°$ C. The density of the ice varied from 0·50 gm. cm.$^{-3}$ near the top surface to 0·80 gm. cm.$^{-3}$ at a depth of 55 m. and then approached the value of solid ice (0·91 gm. cm.$^{-3}$).

The constant A appearing in Eq. (14) can be obtained from the results of Glen on the creep of ice. Glen found that if σ is the uniaxial stress applied to a laboratory specimen then the creep rate is given by

$$K = B \exp(-Q/RT) \exp(Q/RT_m)\sigma^n \qquad (15)$$

where T is the temperature, T_m is the melting point of ice, B is a constant equal to 0·017 bars$^{-4\cdot2}$ years^{-1} (one bar is equal to 10^6 dynes cm.$^{-2}$), Q is an activation energy and is equal to 32,000 cal/mol., n is a constant equal to 4·2, and R is the gas constant. In terms of the quantities entering into Eq. (15) the constant A is given by

$$A^{-n} = (\sqrt{3})^{n+1}2^{-1}B \exp(-Q/RT) \exp(Q/RT_m).$$

(See Nye[1] for the reason for the inclusion of the factor in front of B.)

The term $\int_0^h A\,dy$ in Eq. (14) can be calculated if the temperature is known at all depths. Unfortunately it is not known below 100 m. But enough of the temperature distribution is known so that upper and lower limits can be set on $\int_0^h A\,dy$. If it is assumed that the temperature increases linearly from $-17{\cdot}5°$ C. at the top surface to $-16{\cdot}5°$ C. at a depth of 100 m. and then remains constant at $-16{\cdot}5°$ C. to the bottom surface, Eq. (14) predicts a creep rate of $1{\cdot}2\times10^{-3}$ years^{-1} (with $\rho_W = 1{\cdot}025$ gm. cm.$^{-3}$ and $\rho_I = 0{\cdot}82$ gm. cm.$^{-3}$, value estimated from quoted thickness of shelf and its height above sea level[8]). If it is assumed that the temperature increases linearly from $-16{\cdot}5°$ C. at a depth of 100 m. to $-1{\cdot}5°$ C. at the bottom surface, Eq. (14) predicts a creep rate of $2{\cdot}3\times10^{-3}$ years^{-1}. The observed rate, $1{\cdot}8\times10^{-3}$ years^{-1}, lies between these calculated extreme values. If the average density of the ice is as high as 0·88 gm. cm.$^{-3}$ instead of the estimated value of 0·82 gm. cm.$^{-3}$, the predicted creep rates are reduced by a factor of three. As mentioned previously an exact solution for the creep rate of an ice shelf can be easily obtained for the case where the density of the ice is a function of the depth. Thus if the density-depth curve is known a better check can be obtained on this theory. Another refinement on the calculation of the creep rate can be made if the variation of the constant A with density is known.

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Appendix

Consider a detached floating ice sheet which is free to creep in both the x and z direction. Consider the situation far from the edge of the sheet. Assume again that the various stresses are independent of x and z. Eqs. (2), (3), and (4) will still be valid. It is reasonable in this situation to take σ_{xx} equal to σ_{zz} since an arbitrary rotation of the axes about the y axis ought not to change the form of the solution. Eq. (10) now takes the form

$$K=\dot{\epsilon}_{xx}=\dot{\epsilon}_{zz}=-2\dot{\epsilon}_{yy}=A^n\left|\frac{\sigma_{xx}-\sigma_{yy}}{\sqrt{3}}\right|^{n-1}\left(\frac{\sigma_{xx}-\sigma_{yy}}{3}\right) \quad . \quad . \quad . \quad . \quad (16)$$

Proceeding as before we find for K

$$K=\frac{1}{\sqrt{3}}\left(\frac{\rho_I g h^2}{2\sqrt{3}}\right)^n\left(1-\frac{\rho_I}{\rho_W}\right)^n\Big/\left[\int_0^h A\, dy\right]^n \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The creep rate predicted by Eq. (17) is a factor $(2/\sqrt{3})^n/\sqrt{3} \sim 1\cdot1$ larger than that predicted by Eq. (14). The creep rate predicted by Eq. (17) is isotropic. That is, any base line on the shelf, regardless of its orientation, will stretch at this rate. For Eq. (14) only a base line perpendicular to the sea front will stretch. In any actual situation in the Antarctic the truth probably lies somewhere between Eqs. (14) and (17).

When the density of ice, ρ_I, is a function of the depth, the term $\frac{1}{2}\rho_I h^2(1-\rho_I/\rho_W)$ in Eqs. (14) and (17) should be replaced by

$$\int_0^h\int_x^h \rho_I(y)\, dy\, dx-\frac{1}{2\rho_W}\left[\int_0^h \rho_I(y)\, dy\right]^2.$$